CIVE70095: Structural Dynamics **Autumn 2023/2024**

Lecture 1: Dynamic actions and responses

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Learning outcomes

By the end of this session you should be able to:

- Classify different dynamic loads according to broad qualitative descriptions.
- Appreciate that loads of arbitrary complexity may be modelled as linear sums of sines and cosines.
- Understand what a Fourier Transform is as well as why and how to use it.

1.1 Dynamic loads

Thus far, in your previous studies, you have been mainly concerned with the response of structures subjected to different types of **static loads** (i.e., loads that do not change in time or whose change in time is slow relative to the dynamic properties of the structure). By contrast, the single defining feature of this course, and the one that sets it apart from those that you are already very familiar with, is that the loads we are about to consider are **dynamic loads**. That is, the loads we are interested in this course **vary in time**. This implies that:

- the *amplitude* of the dynamic load may change with time,
- the *position* of the dynamic load may change with time,
- the *direction* of application of a dynamic load may change with time, or,
- some *combination* of these three characteristics may be involved.

Furthermore, given that the loading action is now a function of time, the corresponding structural stresses and strains will also be functions of time. The fact that the position, magnitude and direction of application of dynamic loads vary with time renders our analyses more complex than the static load cases that you are already used to deal with.

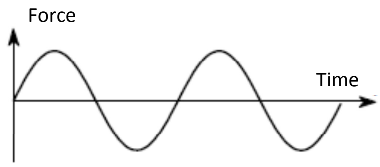
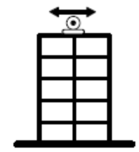
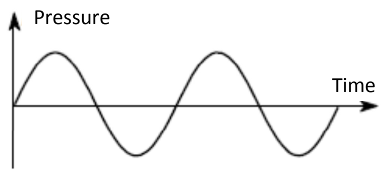
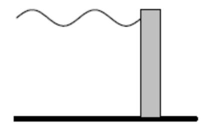
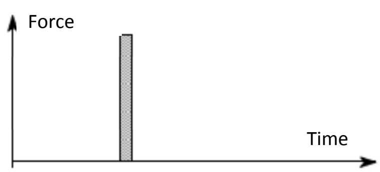
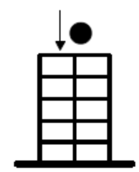
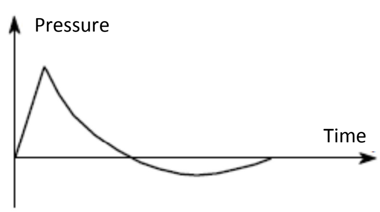
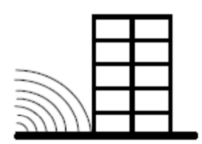
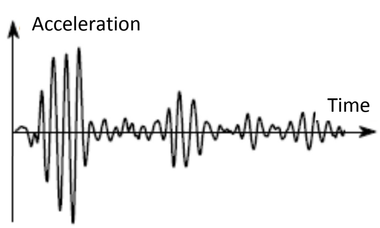
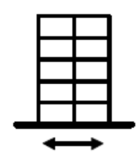
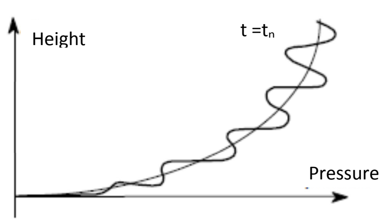
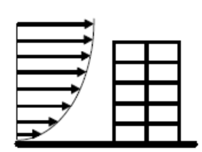
Forced vibration caused by mechanical equipment The effects caused by machines and equipment with rotative components that displace periodically are included within this group.		
Wave (marine) loading Some hydraulic structures are subjected to the (periodic) dynamic effects of hydraulic pressures.		
Impact actions When one mass collides with a structure, it induces an impulsive force (applied to both mass and structure) which causes vibrations.		
Blast An explosion causes air pressure waves (and ground movements) which affect nearby structures.		
Earthquake loading Earthquake ground-motions induce accelerations and displacements that can severely damage structures.		
Wind loading The magnitude of pressures that wind exerts over buildings changes with time. This induces vibrations which may become critical (e.g. fatigue).		

Table 1.1: Examples of dynamic loads. Adapted from [CP93]

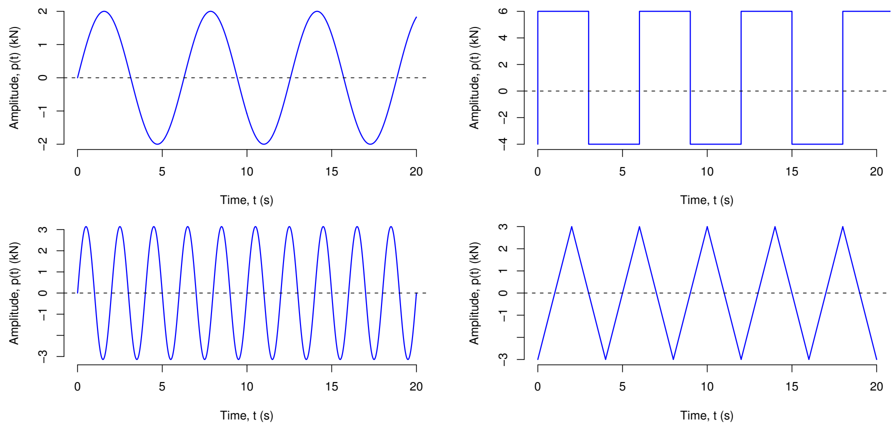
1.1.1 Types of dynamic loads

Table 1.1 presents some examples of the most common types of actions we, as structural engineers, deal with. It can be appreciated from Table 1.1 that most loads can be broadly grouped into two generic categories.

- **Periodic :** such as those caused by rotating machinery or waves
- **Aperiodic :** like the actions caused by earthquakes or blast

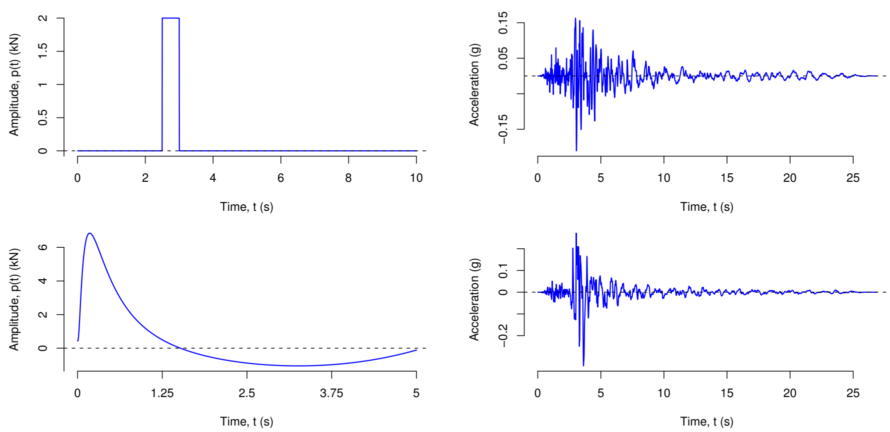
Furthermore, there may be significant differences between the loads considered as periodic. In fact, the group of periodic loads may refer to **harmonic** and **non-harmonic loads**. See Figure 1.1.

Likewise, the group of aperiodic loads includes both **impulsive** loads (such as those arising from blast) and other more **general transient** loads (such as those caused by earthquakes). See Figure 1.2.



(a) Harmonic loads (b) Non-harmonic loads

Figure 1.1: Periodic loads



(a) Impulsive loads (b) Transient (seismic) loads

Figure 1.2: Aperiodic loads

1.1.2 Basic approach in time-domain analysis

The main focus of our course is on **linear elastic structures** (i.e., structures that have not exceeded or are not expected to exceed their yield or elastic limit). That means that if we take a given structure (e.g. a simply supported beam) and apply to it a point load (e.g. in the middle), we will observe a certain deformation which will follow a pattern that we already know from our previous studies. That also means that if we apply a second load (e.g. a distributed load) we will obtain a different displacement profile whose solution will also be known to us from our previous courses. But most importantly, we will recognise that **for a linear structure, the order of application of loads is not important.** And that the **total response is simply the sum** of the two displacement profiles corresponding to each applied load individually. This principle is more formally called **principle of linear superposition**. Very importantly: this idea can be extended to dynamic problems as well! For a structure responding in the linear range, if we apply two sinusoidal loads simultaneously, the total response would be equal to the sum of the individual responses due to each of the sinusoidal loads.

Therefore, for a linear structure, we may determine the response to a **load of arbitrary complexity** by:

1. decomposing the load into a series of harmonics,
2. finding the response of the structure to each individual trigonometric (harmonic) term,
3. then summing all of these responses together to obtain the total response.

1.2 From the Fourier Series to the Fourier Transform

1.2.1 Periodic functions as combinations of sinusoids - Fourier Series

Any periodic function can be represented as a linear combination of sines and cosines. A sine is a function of the form:

$$g(t) = A\sin(2\pi\nu t + \phi) \tag{1.1}$$

where:

- A is the amplitude
- ν is the frequency measured in cycles per second
- ϕ is the phase (responsible for getting values other than 0 at $t = 0$)
- $\omega = 2\pi\nu$ is the circular frequency measured in radians per second
- In addition, we can define period as $T = 1/\nu$, the inverse of the frequency.

A cosine function can be viewed as a shifted sine (e.g. a sine with $\phi = \pi/2$).

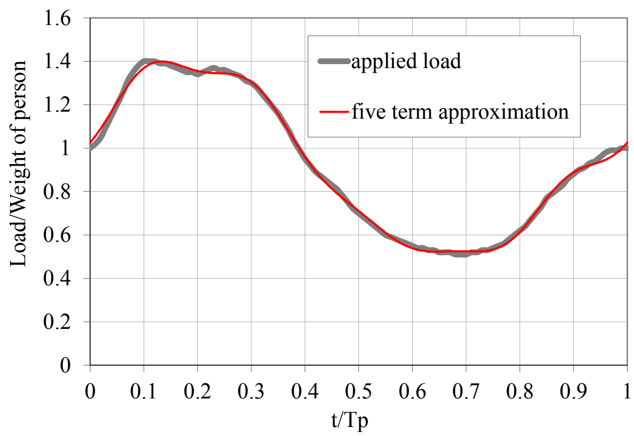
Any periodic function (representing a load or a response) can be approximated by a function $f(t)$:

$$f(t) = A_0 + \sum_{k=1}^{n} \left(A_k \cos\left(2\pi \nu_k t\right) + B_k \sin\left(2\pi \nu_k t\right)\right) \tag{1.2}$$

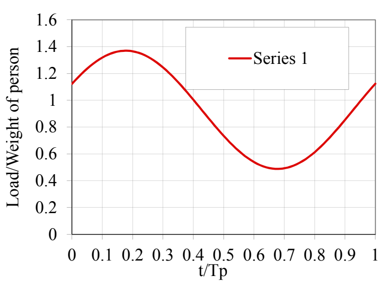
The combination of both sines and cosines, as opposed to just sines, allows for the representation of loads (load functions) with $f(0) \neq 0$ in a simpler way than phase shifting. (e.g. $f(t) = A_0 + \sum_{i=1}^{n} \left(A_i \sin\left(2\pi \nu_k t + \phi_i\right)\right)$). But both are equivalent!

As an example, let's consider the load from a person walking approximated by Equation 1.3 and shown in Figure 1.3.

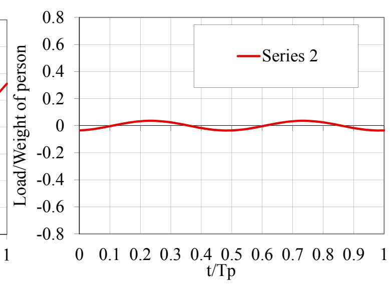
$$\begin{aligned} f_{person}(t) = 0.929 + 0.1943\cos\left(2\pi t/T_p\right) + 0.3954\sin\left(2\pi t/T_p\right) - 0.0352\cos\left(4\pi t/T_p\right) + 0.007\sin\left(4\pi t/T_p\right) \\ -0.0065\cos\left(6\pi t/T_p\right) - 0.0073\sin\left(6\pi t/T_p\right) - 0.0256\cos\left(8\pi t/T_p\right) + 0.0141\sin\left(8\pi t/T_p\right) \\ -0.0304\cos\left(10\pi t/T_p\right) + 0.0028\sin\left(10\pi t/T_p\right) \end{aligned} \tag{1.3}$$



(a) Periodic load of a person walking and its Fourier representation

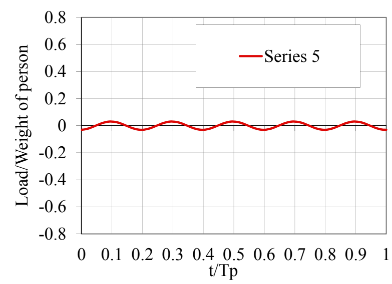


(b) First component



(c) Second component

...



(d) Fifth component

Figure 1.3: Fourier analysis of a periodic load from a person walking

The frequency analysis of $f_{person}(t)$, can be summarized as shown in Table 1.2. This table provides the amplitudes of sine waves and cosine waves corresponding to each frequency of f_{person}.

The representation of a periodic function (load or response) as a linear combination of sines and cosines is known as **Fourier Series Expansion.** This can be represented graphically as depicted in Figure 1.4 for each **harmonic component**.

k	Frequency (ν_k)	Cosine Amplitude (A_k)	Sine Amplitude (B_k)	Amplitude of the harmonic ($\sqrt{A_k^2 + B_k^2}$)
1	$1/T_p$	0.1943	0.3954	0.4406
2	$2/T_p$	-0.0352	0.007	0.0359
3	$3/T_p$	-0.0065	-0.0073	0.0098
4	$4/T_p$	-0.0256	0.0141	0.0292
5	$5/T_p$	-0.0304	0.0028	0.0305

Table 1.2: Fourier analysis of a periodic load from a person walking

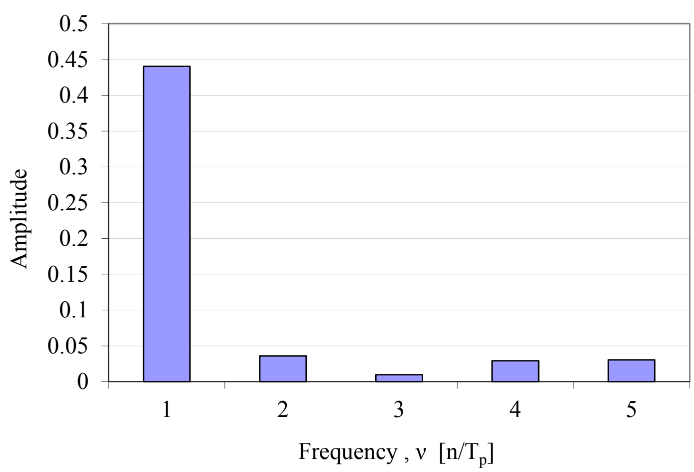
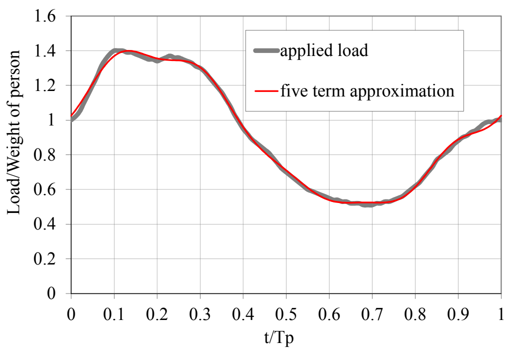


Figure 1.4: Fourier spectrum of a periodic load from a person walking

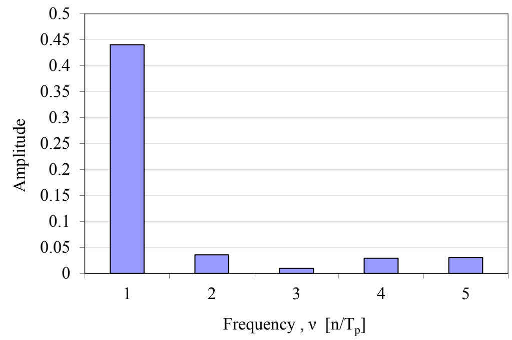
1.2.2 Non-periodic functions as combinations of harmonics - Fourier Transform

We have just seen how the Fourier Series expansion helps us to characterise a periodic function (e.g. pedestrian load) in terms of its frequency components and their amplitude. The **Fourier Transform** is a tool for obtaining such frequency and amplitude information for sequences and **functions that are not obviously periodic or are decidedly aperiodic.** Take for example the history of accelerations recorded during the Mexico Earthquake of 1985, a typical example of a transient load. Its corresponding Fourier Spectrum is presented in Figure 1.5. It can be observed from this figure that the maximum amplitude takes place at a period close to 2 seconds and that the amplitudes for longer periods are very low. This is peculiar to this record and is due to the site amplification caused by the soft soil conditions and configuration of the Mexico City area.

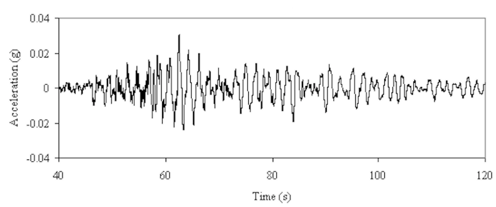
The **Fourier Transform** is a mathematical tool that takes a function in time, measures every harmonic component, and returns the information in terms of frequencies.
Or
Given the smoothie, it finds the recipe!



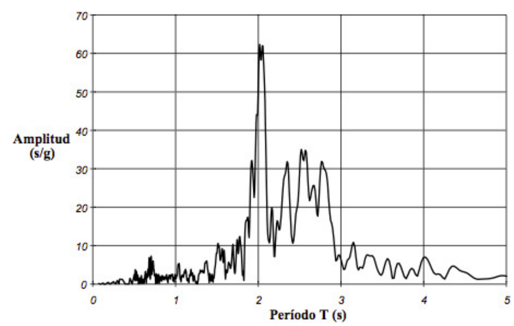
(a) Periodic load of a person walking



(b) Fourier expansion of the periodic load on the left



(c) Accelerogram from the Mexico City Earthquake, 1985



(d) Fourier Spectrum of the accelerogram on the left

Figure 1.5: Fourier expansion and Fourier spectrum

We define the Fourier Transform (FT) and its Inverse (IFT) as:

The **Fourier Transform** of a function in the time domain $f(t)$ is defined as:

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \nu t} dt \tag{1.4}$$

while the **Inverse Fourier Transform** of a function in the frequency domain $F(\nu)$,is:

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu \tag{1.5}$$

Equation 1.5 is the continuous generalization of expressing $f(t)$ as a combination of sinusoids, as will be discussed below. This expression is known as the *Inverse Fourier Transform*. On the other hand, Equation 1.4 provides the means for finding the amplitude for each frequency ν, given that the integral indeed converges. The result of applying the Fourier Transform to a function is called *frequency spectrum* or in short the spectrum, as we have seen before. Note that some texts may use slightly different definitions of FT and IFT.

1.2.3 Mathematical background for the Fourier Transform - Complex numbers and exponentials

The best way to understand Equations 1.4 and 1.5 is to see their relation to Equation 1.2. To this end, it is important to recall that another way of writing sinusoids, such as those presented in Equation 1.2, relies on the following complex exponential equalities:

$$e^{i\theta} = cos\theta + isin\theta \qquad e^{-i\theta} = cos\theta - isin\theta \tag{1.6}$$

where i is the square root of -1. Through addition and subtraction, Equations 1.6 can be rewritten as:

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.7}$$

Therefore, by substituting Equations 1.6 into Equation 1.2:

$$f(t) = A_0 + \sum_{k=1}^{n} \left[\frac{A_k}{2} \left(e^{2\pi i \nu_k t} + e^{-2\pi i \nu_k t} \right) + \frac{B_k}{2i} \left(e^{2\pi i \nu_k t} - e^{-2\pi i \nu_k t} \right) \right] \tag{1.8}$$

if we denote:

$$\begin{aligned} C_k &= \frac{A_k - iB_k}{2} \quad &;& \quad k > 0 \\ C_k &= \frac{A_k + iB_k}{2} \quad &;& \quad k < 0 \\ C_k &= 0 \quad &;& \quad k = 0 \\ \nu_k &= -\nu_{-k} \quad &;& \quad k < 0 \end{aligned} \tag{1.9}$$

Equation 1.8 can be rewritten as:

$$f(t) = A_0 + \sum_{k=-n}^{n} \left[C_k e^{2\pi i \nu_k t} \right] \tag{1.10}$$

Using this new notation, the frequency analysis of Table 1.2 can be reformulated as:

k	Frequency (ν_k)	C_k
-5	$-5/T_p$	-0.0152 +i 0.0014
-4	$-4/T_p$	-0.0128 +i0.00705
-3	$-3/T_p$	-0.00325 -i0.00365
-2	$-2/T_p$	-0.0176 +i0.0035
-1	$-1/T_p$	0.09715 +i0.1977
0	0	0
1	$1/T_p$	0.09715 -i0.1977
2	$2/T_p$	-0.0176 -i0.0035
3	$3/T_p$	-0.00325 +i0.00365
4	$4/T_p$	-0.0128 -i0.00705
5	$5/T_p$	-0.0152 -i0.0014

Table 1.3: Fourier analysis of a periodic load from a person walking. Alternative formulation.

A further manipulation of Equation 1.6 can be done by employing the polar notation of complex numbers as:

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{1.11}$$

where:

$$r = |x + iy| = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x} \tag{1.12}$$

using this representation, we can get:

$$C_k e^{2\pi i \nu_k t} = r_k e^{i\phi_k} e^{2\pi i \nu_k t} = r_k e^{i(2\pi\nu_k t + \phi_k)} \tag{1.13}$$

where:

$$r_k = \left(\frac{A_k^2 + B_k^2}{4}\right)^{1/2} \quad \text{and} \quad \tan\phi_k = \begin{cases} \dfrac{-B_k}{A_k} & k > 0 \\ \dfrac{B_k}{A_k} & k < 0 \end{cases} \tag{1.14}$$

From all the above we obtain:

$$f(t) = A_0 + \sum_{k=-n}^{n} r_k e^{i(2\pi\nu_k t + \phi_k)} \tag{1.15}$$

where ν_k is the k^{th} frequency, r_k is the amplitude and ϕ_k is the phase.

If we now generalize the summation in Equation 1.15 for a continuous function, we get Equation 1.5:

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu$$

By the same token we can obtain Equation 1.4.

Table 1.4 lists some properties of the Fourier transform which make it useful in practice.

Property	$f(t)$	$F(\nu)$
1. Linearity	$af_1(t) + bf_2(t)$	$aF_1(\nu) + bF_2(\nu)$
2. Convolution theorem	$f_1(t) * f_2(t)$	$F_1(\nu)F_2(\nu)$
3. Product theorem	$f_1(t)f_2(t)$	$F_1(\nu) * F_2(\nu)$
4. Time shifting	$f(t - t_0)$	$F(\nu)e^{-2\pi i \nu t_0}$
5. Frequency shifting	$f(t)e^{-2\pi i \nu_0 t}$	$F(\nu - \nu_0)$
6. Scaling	$f(at)$	$\lvert a\rvert^{-1} F(\nu/a)$

Table 1.4: Fourier analysis of a periodic load from a person walking. Alternative formulation

Properties 2 and 3 in Table 1.4 state that convolution in the time domain corresponds to multiplication of the coefficients in the frequency domain, and vice versa. This is one of the most useful properties of the Fourier Transform. Property 4 in Table 1.4 states that shifting of the original function in time corresponds to a change of phase of the sinusoids comprising the function. Similarly, a sinusoidal modulation in the function corresponds to a phase shift in the frequency (Property 5).

Example 1:

Let's introduce the *Dirac Delta* function $\delta(t)$ such that:

- **Dirac delta function:**
  - Definition

$$\delta(t-\tau) = \begin{cases} 0 & \text{for} \quad t \neq \tau \\ \infty & \text{for} \quad t = \tau \end{cases}$$

  - Properties

$$\int_{-\infty}^{\infty} \delta(t-\tau) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t) dt = f(\tau)$$

- Now let's find the Fourier Transform of a Delta function:

$$F(\nu) = \int_{-\infty}^{\infty} \delta(t-a) e^{-2\pi i \nu t} dt$$

recalling the properties of the Delta function as explained above, we can see that it can be understood as an operator that when multiplied with a given function it outputs the value of the function at the location of the delta, e.g. $\int_{-\infty}^{\infty} \delta(t-\tau) f(t) dt = f(\tau)$. Then:

$$F(\nu) = \int_{-\infty}^{\infty} \delta(t-a) e^{-2\pi i \nu t} dt = e^{-2\pi i \nu a}$$

- That means that the Inverse Fourier Transform of a complex exponential is an impulse function:

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t} d\nu = \int_{-\infty}^{\infty} e^{-2\pi i \nu a} e^{2\pi i \nu t} d\nu$$

$$f(t) = \delta(t-a) = \int_{-\infty}^{\infty} e^{2\pi i \nu (t-a)} d\nu$$

Example 2:

Let $f(t)$ be a harmonic load of the form: $f(t) = \cos(2\pi at)$. Find its Fourier transform.

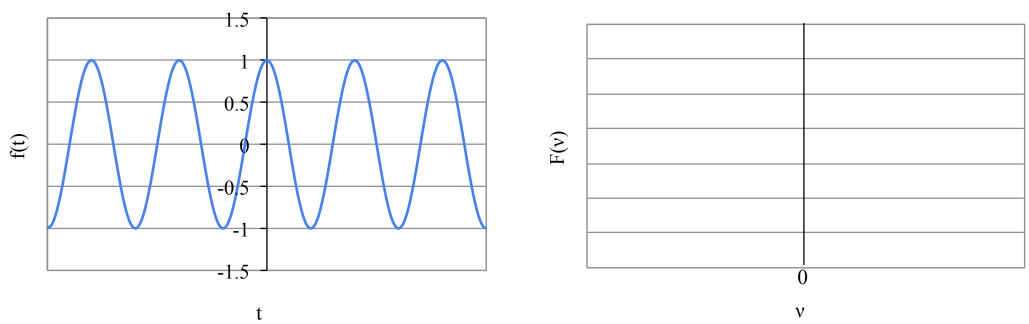


Figure 1.6: $f(t) = cos(2\pi at)$ (left) and its Fourier transform (right) - *Complete in class*

1.3 The Discrete Time Fourier Transform

1.3.1 Discrete signals

A number of signals are continuous and have a continuous range of frequencies and the functions that we employ to model them can be continuous as well. However, more often than not, the function describing a phenomena (load or response) is unknown. In those cases, experimental data needs to be gathered and analysed, which is usually done by measuring **discrete values at various points in time**. This data will be analysed by **digital computers** and we should be mindful that the computers employed in our everyday engineering calculations work inherently with discrete variables (bits) after all. Therefore, for all practical purposes we will be more interested in a Fourier Transform in the **discrete time** and **discrete frequency** domains.

1.3.2 Discrete Time Fourier Transform

The Discrete Fourier Transform (DFT) maps a discrete periodic series $x[n]$, where n is an integer and the period is N to another function $X(e^{i\omega})$ of frequency coefficients. In fact, our previous discussion has highlighted that the Fourier series coefficients can be viewed as samples of an envelope function and that as the period of the function increases, the samples become more and more finely spaced. Using the same principle, **which will be discussed in more detail in class**, we can arrive to the following pair of equations for the Discrete-time Fourier Transform:

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} \tag{1.16}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{i\omega})e^{i\omega n}d\omega \tag{1.17}$$

1.3.3 Fast Fourier Transform

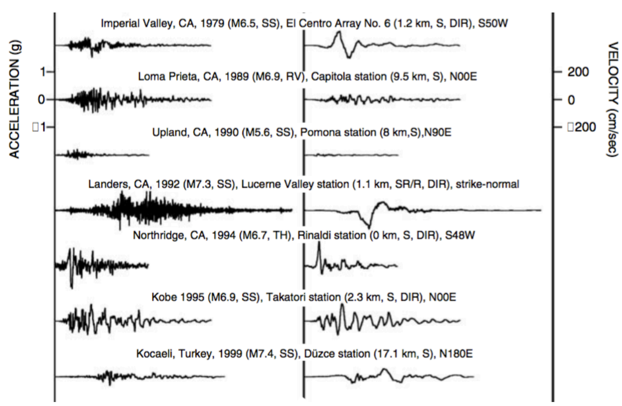
But the success and power of the Fourier transform would have stalled were it not for an ingenious algorithm that would make it easy for digital computers to compute the DFT. In fact, the publication of Cooley and Tukey's algorithm [CT65] in 1965 was a turning point in digital signal processing and numerical analysis. This is probably the reason for the ubiquity of Fourier analysis in nearly all aspects of human endeavour today, from music and video processing to structural analysis. In essence, Cooley and Tukey showed that the DFT, previously thought to require N^2 arithmetic operations consuming hundreds of computer hours could in fact be calculated much quicker by their new *Fast Fourier Transform* algorithm using only $N \log N$ operations. We do not have time as part of this course to delve into the fascinating intricacies of FFT computations but there are many good resources online to which you can refer in case of personal interest, including the excellent lecture by Prof Demaine: `https://www.youtube.com/watch?v=iTMn0Kt18tg`.

Appendix: Code representation of dynamic loads

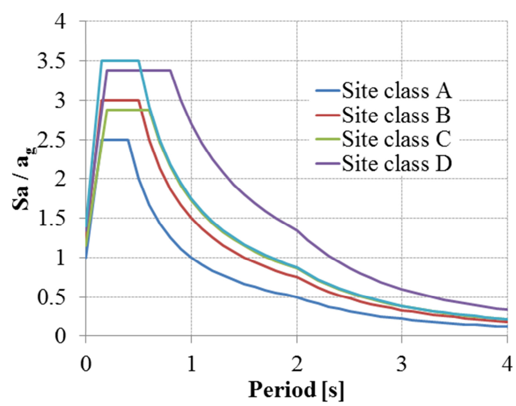
It is possible to formulate a full mathematical description of some forms of periodic loading (like rotating machinery). On the other hand, the three most common natural dynamic loads that we will encounter are earthquake, wind and wave loading for which a full description is very difficult. Instead codes of practice assume very simple descriptions of these loading actions.

The loads associated with **wind** are applied directly to the structure in the form of pressures and are characterized in terms of a wind velocity. The pressure distributions encountered in reality depend upon the specific geometry of the structure being considered as well as factors related to the uniformity and predominant direction of the wind field as well as interactions with its surroundings. Given that such effects may very quickly become complex, codes define conservative load factors as a surrogate for the calculation of equivalent loads.

Something similar happens in the case of **earthquake** loading. As we know, earthquakes by themselves (Figure 1.7a) do not result in the direct application of loads upon a structure, rather they result in strong **ground-motions** that shake structures. Those support movements will in turn produce relative accelerations in the building leading to inertial loads within the structure itself, as we will discuss further in the next class. To avoid dynamic calculations, conventional code-based approaches use equivalent static procedures that define earthquake loads to be directly applied to the structure in a static manner. Numerous assumptions are involved in the process and it is important to have them present when conducting code-based analyses. In Eurocode 8 [EC08], earthquake actions are prescribed in terms of an elastic response spectra. These spectra (given in Figure 1.7b) allows us to determine the level of equivalent lateral earthquake load for a particular structure characterized primarily by its first natural period. We will explore what a response spectrum is and how is it obtained later in the course. It must be kept in mind, however, that the loading cannot be prescribed in terms of such generic static representations in all cases and that in order to perform more realistic structural analyses we must use acceleration time-histories directly imputed into our models, such as those shown in Figure 1.7a. The issue then becomes how to account for the variability in the ground-motion characteristics.



(a) Acceleration series of some famous earthquakes plotted at the same scale [BC04]



(b) Eurocode 8 Seismic Design Spectra

Figure 1.7: Code representation of seismic loads

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