Structural Dynamics

Fourier Transform of a Train of Deltas

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A train of deltas is mathematically known as a *comb function* $III(t)$ and is defined as:

$$III(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{1}$$

which is formed by a series of delta functions spaced at T intervals (periodic).

The first step towards obtaining its Fourier Transform is to prove that:

$$III(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nt/T} \tag{2}$$

To this end we have to remember that any periodic signal can be decomposed into a series of the form:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nt/T} \tag{3}$$

where

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i2\pi nt/T} dt \tag{4}$$

This is in fact similar to the Fourier Transform but only integrating on one period of $f(t)$ which *is periodic*. Therefore we can compute C_n such that:

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} III(t) e^{-i2\pi nt/T} dt \tag{5}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} III(t) e^{-i2\pi nt/T} dt \tag{6}$$

$$C_n = \frac{1}{T} \tag{7}$$

Now we can compute the Fourier Transform of the Dirac Delta Comb by considering that we can invert the order of $\int$ and $\sum$, so:

$$F(\nu) = \int_{-\infty}^{\infty} \left(\sum_{-\infty}^{\infty} \frac{1}{T} e^{i2\pi nt/T} \right) e^{-i2\pi \nu t} dt \tag{8}$$

$$F(\nu) = \sum_{-\infty}^{\infty} \frac{1}{T} F(e^{i2\pi nt/T})(2\pi\nu) \tag{9}$$

$$F(\nu) = \frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta\left(2\pi\nu - \frac{2\pi n}{T}\right) \tag{10}$$

which is the same as:

$$F(\nu) = \frac{1}{T} \sum_{-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right) \tag{11}$$

which is another Delta Comb but this time in the frequency domain!